Constraints and gauge

① Unconstrained Lagrangian & Hamiltonian systems

$$\mathcal{L} = \int L(q, \dot{q})\, dt$$

q - generalized position

$\dot{q}$ - " " velocity

$\delta \mathcal{L} = 0 \Rightarrow$ Euler-Lagrange eqs

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^n}\right) - \frac{\partial L}{\partial q^n} = 0$$

Ex q is position x of a particle with mass m

$$L = \tfrac{1}{2} m \dot{x}^2 - V(x)$$

EL eqs: $\quad m\ddot{x} = -\dfrac{\partial V}{\partial x}$

Question: What eqs. of motion can be put in Lagrangian form (aka "the inverse problem") ?

Legendre transform:

$$(q, \dot{q}) \longrightarrow (q, p)$$

$$p_n := \frac{\partial L}{\partial \dot{q}^n}$$

Hamiltonian:

$$H := \sum_n p_n \dot{q}^n - L$$

Hamilton's eqs:

$$\dot{p}_n = -\frac{\partial H}{\partial q^n}, \quad \dot{q}^n = \frac{\partial H}{\partial p_n}$$

Ex (as above)

$$L = \frac{1}{2} m \dot{x}^2 - V(x)$$

$$p = m\dot{x}$$

$$H = m\dot{x}^2 - \frac{1}{2} m\dot{x}^2 + V(x)$$

$$= \frac{1}{2} m\dot{x}^2 + V(x)$$

$$= \frac{1}{2} \frac{p^2}{m} + V(x) \quad (\text{Energy})$$

$$\dot{p} = m\ddot{x} = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x} \quad ✓$$

② Constrained Hamiltonian systems

Expand EL eqs

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^n}\right) - \frac{\partial L}{\partial q^n} = 0$$

$$\ddot{q}^m \underbrace{\left(\frac{\partial^2 L}{\partial \dot{q}^m \partial \dot{q}^n}\right)}_{\substack{W_{mn} \\ \text{(Hessian)}}} = \frac{\partial L}{\partial q^n} - \dot{q}^m \left(\frac{\partial^2 L}{\partial q^m \partial \dot{q}^n}\right)$$

When $\det(W_{mn}) = 0$ can't solve for $\ddot{q}^m$ in terms of q^n and $\dot{q}^n$ -- apparent breakdown in determinism.

Also get Hamiltonian constraints

Find that the canonical momenta

$$p_n := \frac{\partial L}{\partial \dot{q}^n}$$

are not independent but must satisfy

$$\phi_i(p, q) = 0 \quad , \; i = 1, 2, \ldots, N$$

called the <u>primary</u> <u>constraints</u>.

<u>Secondary</u> <u>constraints</u> may emerge from the requirement that the primary constraints be preserved by the evolution.

Ex 1 $L = \frac{1}{2}(\dot{q}^1 - \dot{q}^2)^2$

$$p_1 = \dot{q}^1 - \dot{q}^2$$

$$p_2 = \dot{q}^2 - \dot{q}^1$$

Primary constraint:

$$\phi = p_1 + p_2 = 0$$

See Fig. 1 of Henneaux & Teitelboim

EL eqs: $\ddot{q}^1 - \ddot{q}^2 = 0$

$$\ddot{q}^2 - \ddot{q}^1 = 0$$

If $q^1(t), q^2(t)$ are sols. then so are

$$\tilde{q}^{1,2}(t) = q^{1,2}(t) + \underbrace{f(t)}_{\text{arbitrary function of } t}$$

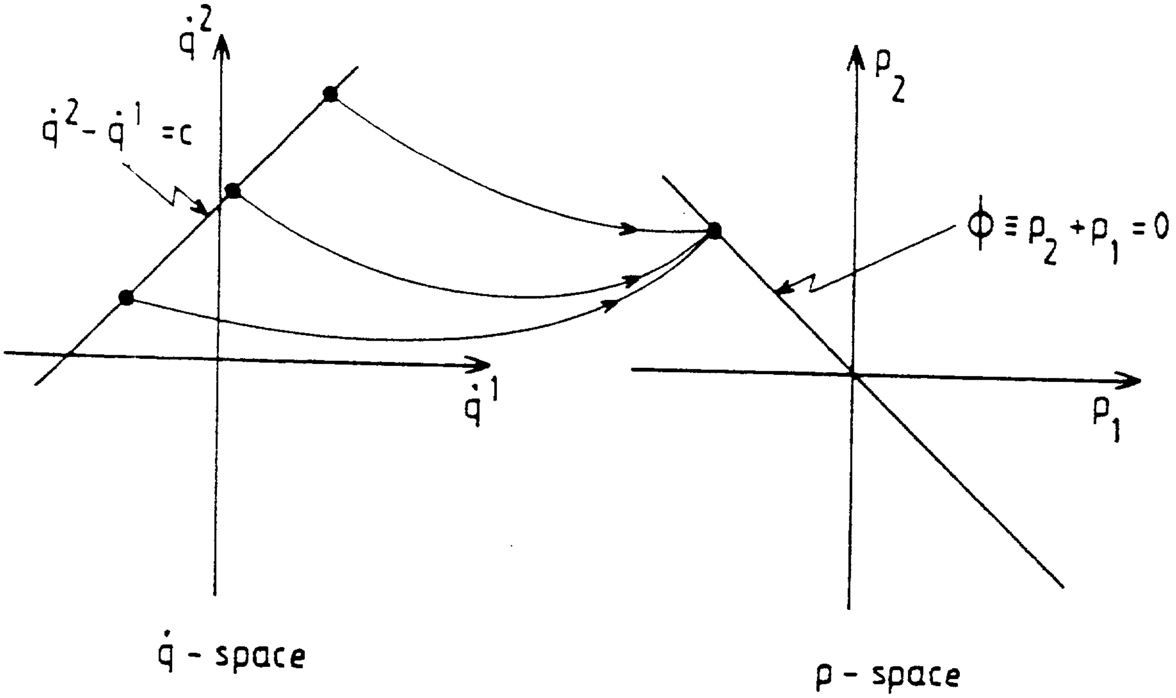


Figure 1: The figure shows the example of a system with two q's and Lagrangian $\frac{1}{2}(\dot{q}^1 - \dot{q}^2)^2$. The momenta are $p_1 = \dot{q}^1 - \dot{q}^2$ and $p_2 = \dot{q}^2 - \dot{q}^1$. There is one primary constraint $\phi = p_1 + p_2 = 0$. All of $\dot{q}$-space is mapped on the straight line $p_1 + p_2 = 0$ of p-space. Moreover, all the $\dot{q}$'s on the straight line $\dot{q}^2 - \dot{q}^1 = c$ are mapped on the same point $p_1 = -c = -p_2$ belonging to the constraint surface $\phi = 0$. The transformation $\dot{q} \rightarrow p$ is thus neither one-to-one nor onto. To render the transformation invertible, one needs to adjoin extra parameters to the p's (see below).

The presence of the arb. function of t seems to mean that determinism fails.

But can reason the other way round: determinism *must* hold; the apparent failure can be sopped up by gauge freedom.

See Henneaux & Teitelboim, pp. 16-17

The presence of arbitrary functions v^a in the total Hamiltonian tells us that not all the q's and p's are observable. In other words, although the physical state is uniquely defined once a set of q's and p's is given, the converse is not true—*i.e.*, there is more than one set of values of the canonical variables representing a given physical state. To see how this conclusion comes about, we notice that if we give an initial set of canonical variables at the time t_1 and thereby completely define the physical state at that time, we expect the equations of motion *to fully determine the physical state at other times.* Thus, by definition, any ambiguity in the value of the canonical variables at $t_2 \neq t_1$ should be a physically irrelevant ambiguity.

Ex 2 Maxwellian st.
Want a L that is invariant under the st symmetries:

$$\underset{\sim}{x} \Rightarrow \underset{\sim}{x}' = \underset{\nearrow}{R}\,\underset{\sim}{x} + \underset{\sim}{f}\underset{\nwarrow}{(t)}$$

const. rotation

arb. function of t

$$L = \sum_{j<k} \frac{m_j m_k}{2\, m_{tot}} (\dot{\underset{\sim}{x}}_j - \dot{\underset{\sim}{x}}_k)^2 - V(|\underset{\sim}{x}_j - \underset{\sim}{x}_k|)$$

$$m_{tot} := \sum_i m_i$$

$$p_i^\alpha = \frac{\partial L}{\partial \dot{x}_i^\alpha} = m_i \dot{x}_i^\alpha - \frac{m_i}{m_{tot}} \sum_k m_k \dot{x}_k^\alpha$$

Primary constraint : $\sum_i p_i^\alpha = 0$, $\alpha = 1, 2, 3$

EL eqs:

$$\frac{d}{dt}\left(m_i\left(\dot{x}_i^\alpha - \frac{1}{m_{tot}}\sum_k m_k \dot{x}_k^\alpha\right)\right) = -\frac{\partial V}{\partial x_i^\alpha}$$

If $x_i^\alpha(t)$ is a solution, then so is

$$\tilde{x}_i^\alpha(t) = x_i^\alpha(t) + \underbrace{f^\alpha(t)}_{\text{arb. function of } t}$$

Again, the apparent indeterminism is sopped up by gauge freedom.

$*$ This apparatus produces an account of gauge

The primary and secondary constraints together define a subspace

$$\mathfrak{S} \subset \Gamma(q,p)$$

of the phase space Γ, called the constraint surface.

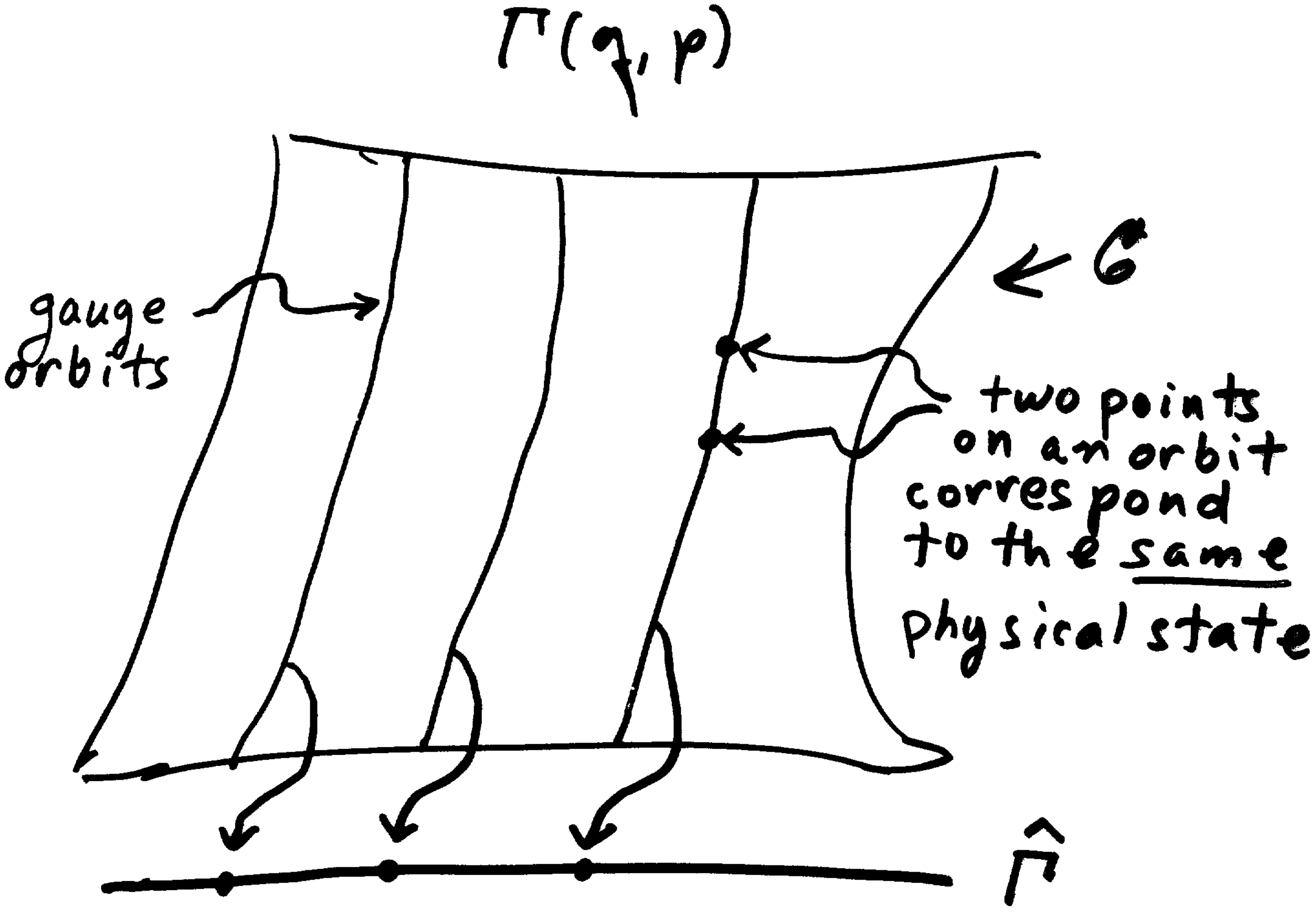
<u>1st class constraints</u> are those Poisson bracket with every constraint vanishes weakly (i.e. on $\mathfrak{S}$)

$$[F,G] := \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}$$

"Observables"

- Functions $F : \mathcal{C} \to \mathbb{R}$ that have vanishing PB with all first class constraints
- Functions on the reduced phase space $\hat{\Gamma}$ obtained by quotienting out $\mathcal{C}$ by the gauge orbits

Based on Dirac's proposal: gauge transformations are generated by the first class constraints.



Each gauge orbit corresponds to a single point of $\hat{\Gamma}$

Modulo tech. difficulties, have normal, unconstrained Hamiltonian mechanics on $\hat{\Gamma}$

Observables evolve deterministically.

__Ex 1__ (again)

q^1 and q^2 __not__ observables:

$[q^1, p_1 + p_2] \neq 0 \neq [q^2, p_1 + p_2]$

__But__ $q^1 - q^2$ __is__ an observable:

$[q^1 - q^2, p_1 + p_2] = 0$

$q^1 - q^2$ __does__ evolve deterministically

__Ex 2__ (Maxwellian case)

similar

③ Hamiltonian formalism for GR

$q \rightsquigarrow h_{ab}$ space metric on a slice Σ

$p \rightsquigarrow \pi^{ab}$ conjugate momentum defined in terms of extrinsic curvature of Σ

Find that GR is a constrained Hamiltonian system

Two families of constraints

$\mathcal{P}_a(h_{ab}, \pi^{cd}) = 0$ Momentum constraint

$\mathcal{H}_0(h_{ab}, \pi^{cd}) = 0$ Hamiltonian constraint

Hamiltonian for GR:

$$H = \int (N\mathcal{H}_0 + N^a \mathcal{P}_a)$$

N - "lapse function"

N^a - "shift functions"

- How do the constraints correspond to the diffeo freedom of GR?

On the constraint surface of the (h_{ab}, π^{cd}) phase space, the canonical transformations generated by

$$\int \xi^a P_a \quad \text{and} \quad \int \xi^0 \mathcal{H}_0$$

correspond to spacetime diffeomorphisms (Unruh & Wald 1989).

- What to make of the notion that motion is pure gauge?

④ What is an "observable" in GR?

"Observable" = gauge invariant quantity

Take the diff(M) group to be gauge for GR

1) No local quantity $Q(p)$ that is a function of spacetime points $p \in M$ is an "observable" -- not even "scalar invariants" -- unless $Q(p) = \text{const}$.

2) Global quantities like

$$\int_M R \sqrt{-g}\, d^4x \quad (\text{if } \int \text{converges})$$

are "observables." But they are not useful for doing physics.

3) Coincidence quantities

Einstein's (1916) escape from the hole argment -- "point coincidences"

Generalize this notion using ideas of Kretchmann (1918) and Komar (1950s).

In a generic solution of the vacuum EFE, M, g_{ab}, the metric will not have any symmetries.

So can find scalar fields ϕ^A, $A = 1, 2, 3, 4$ s.t.

$$\forall p, q \in M \quad p = q \Longleftrightarrow \phi^A(p) = \phi^A(q)$$

Note: The ϕ^A are not "observables" But they can be used to define observables

E.g.

$$g^{AB}(\varphi^C) := \frac{\partial \varphi^A}{\partial x^m} \frac{\partial \varphi^B}{\partial x^n} g^{mn}$$

This functional is on "observable"

How to parse and measure such "observables" ?

knowledge, there is a well-known physical fact which favours an extension of the theory of relativity. Let K be a Galilean system of reference, i.e. a system relatively to which (at least in the four-dimensional region under consideration) a mass, sufficiently distant from other masses, is moving with uniform motion in a straight line. Let K′ be a second system of reference which is moving relatively to K in *uniformly accelerated* translation. Then, relatively to K′, a mass sufficiently distant from other masses would have an accelerated motion such that its acceleration and direction of acceleration are independent of the material composition and physical state of the mass.

Does this permit an observer at rest relatively to K′ to infer that he is on a "really" accelerated system of reference? The answer is in the negative; for the above-mentioned relation of freely movable masses to K′ may be interpreted equally well in the following way. The system of reference K′ is unaccelerated, but the space-time territory in question is under the sway of a gravitational field, which generates the accelerated motion of the bodies relatively to K′.

This view is made possible for us by the teaching of experience as to the existence of a field of force, namely, the gravitational field, which possesses the remarkable property of imparting the same acceleration to all bodies.* The mechanical behaviour of bodies relatively to K′ is the same as presents itself to experience in the case of systems which we are wont to regard as "stationary" or as "privileged." Therefore, from the physical standpoint, the assumption readily suggests itself that the systems K and K′ may both with equal right be looked upon as "stationary," that is to say, they have an equal title as systems of reference for the physical description of phenomena.

It will be seen from these reflexions that in pursuing the general theory of relativity we shall be led to a theory of gravitation, since we are able to "produce" a gravitational field merely by changing the system of co-ordinates. It will also be obvious that the principle of the constancy of the velocity of light *in vacuo* must be modified, since we easily recognize that the path of a ray of light with respect to K′ must in general be curvilinear, if with respect to K light is propagated in a straight line with a definite constant velocity.

§ 3. The Space-Time Continuum. Requirement of General Co-Variance for the Equations Expressing General Laws of Nature

In classical mechanics, as well as in the special theory of relativity, the co-ordinates of space and time have a direct physical meaning. To say that a point-event has the X_1 co-ordinate x_1 means that the projection of the point-event on the axis of X_1, determined by rigid rods and in accordance with the. rules of Euclidean geometry, is obtained by measuring off a given rod (the unit of length) x_1 times from the origin of co-ordinates along the axis of X_1. To say that a point-event has the X_4 co-ordinate $x_4 = t$, means that a standard clock, made to measure time in a definite unit period, and which is stationary relatively to the system of co-ordinates and practically coincident in space with the point-event,* will have measured off $x_4 = t$ periods at the occurrence of the event.

This view of space and time has always been in the minds of physicists, even if, as a rule, they have been unconscious of it. This is clear from the part which these concepts play in physical measurements; it must also have underlain the reader's reflexions on the preceding paragraph (§ 2) for him to connect any meaning with what he there read. But we shall now show that we must put it aside and replace it by a more general view, in order to be able to carry through the postulate of general relativity, if the special theory of relativity applies to the special case of the absence of a gravitational field.

In a space which is free of gravitational fields we introduce a Galilean system of reference K (x, y, z, t), and also a system of co-ordinates K′ (x', y', z', t) in uniform rotation relatively to K. Let the origins of both systems, as well as their axes

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* Eötvös has proved experimentally that the gravitational field has this property in great accuracy.

* We assume the possibility of verifying "simultaneity" for events immediately proximate in space, or—to speak more precisely—for immediate proximity or coincidence in space-time, without giving a definition of this fundamental concept.

of Z, permanently coincide. We shall show that for a space-time measurement in the system K′ the above definition of the physical meaning of lengths and times cannot be maintained. For reasons of symmetry it is clear that a circle around the origin in the X, Y plane of K may at the same time be regarded as a circle in the X′, Y′ plane of K′. We suppose that the circumference and diameter of this circle have been measured with a unit measure infinitely small compared with the radius, and that we have the quotient of the two results. If this experiment were performed with a measuring-rod at rest relatively to the Galilean system K, the quotient would be π. With a measuring-rod at rest relatively to K′, the quotient would be greater than π. This is readily understood if we envisage the whole process of measuring from the "stationary" system K, and take into consideration that the measuring-rod applied to the periphery undergoes a Lorentzian contraction, while the one applied along the radius does not. Hence Euclidean geometry does not apply to K′. The notion of co-ordinates defined above, which presupposes the validity of Euclidean geometry, therefore breaks down in relation to the system K′. So, too, we are unable to introduce a time corresponding to physical requirements in K′, indicated by clocks at rest relatively to K′. To convince ourselves of this impossibility, let us imagine two clocks of identical constitution placed, one at the origin of co-ordinates, and the other at the circumference of the circle, and both envisaged from the "stationary" system K. By a familiar result of the special theory of relativity, the clock at the circumference—judged from K—goes more slowly than the other, because the former is in motion and the latter at rest. An observer at the common origin of co-ordinates, capable of observing the clock at the circumference by means of light, would therefore see it lagging behind the clock beside him. As he will not make up his mind to let the velocity of light along the path in question depend explicitly on the time, he will interpret his observations as showing that the clock at the circumference "really" goes more slowly than the clock at the origin. So he will be obliged to define time in such a way that the rate of a clock depends upon where the clock may be.

We therefore reach this result:—In the general theory of relativity, space and time cannot be defined in such a way that differences of the spatial co-ordinates can be directly measured by the unit measuring-rod, or differences in the time co-ordinate by a standard clock.

The method hitherto employed for laying co-ordinates into the space-time continuum in a definite manner thus breaks down, and there seems to be no other way which would allow us to adapt systems of co-ordinates to the four-dimensional universe so that we might expect from their application a particularly simple formulation of the laws of nature. So there is nothing for it but to regard all imaginable systems of co-ordinates, on principle, as equally suitable for the description of nature. This comes to requiring that:—

The general laws of nature are to be expressed by equations which hold good for all systems of co-ordinates, that is, are co-variant with respect to any substitutions whatever (generally co-variant).

It is clear that a physical theory which satisfies this postulate will also be suitable for the general postulate of relativity. For the sum of *all* substitutions in any case includes those which correspond to all relative motions of three-dimensional systems of co-ordinates. That this requirement of general co-variance, which takes away from space and time the last remnant of physical objectivity, is a natural one, will be seen from the following reflexion. All our space-time verifications invariably amount to a determination of space-time coincidences. If, for example, events consisted merely in the motion of material points, then ultimately nothing would be observable but the meetings of two or more of these points. Moreover, the results of our measurings are nothing but verifications of such meetings of the material points of our measuring instruments with other material points, coincidences between the hands of a clock and points on the clock dial, and observed point-events happening at the same place at the same time.

The introduction of a system of reference serves no other purpose than to facilitate the description of the totality of such coincidences. We allot to the universe four space-time variables x_1, x_2, x_3, x_4 in such a way that for every point-event there is a corresponding system of values of the variables $x_1 \dots x_4$. To two coincident point-events there corresponds one system of values of the variables $x_1 \dots x_4$, i.e. coincidence is characterized by the identity of the co-ordinates. If, in place of the variables $x_1 \dots x_4$, we introduce functions of them, x'_1, x'_2, x'_3, x'_4, as a new system of co-ordinates, so that the systems of values are made to correspond to one another without ambiguity, the equality of all four co-ordinates in the new system will also serve as an expression for the space-time coincidence of the two point-events. As all our physical experience can be ultimately reduced to such coincidences, there is no immediate reason for preferring certain systems of co-ordinates to others, that is to say, we arrive at the requirement of general co-variance.

§ 4. The Relation of the Four Co-ordinates to Measurement in Space and Time

It is not my purpose in this discussion to represent the general theory of relativity as a system that is as simple and logical as possible, and with the minimum number of axioms; but my main object is to develop this theory in such a way that the reader will feel that the path we have entered upon is psychologically the natural one, and that the underlying assumptions will seem to have the highest possible degree of security. With this aim in view let it now be granted that:—

For infinitely small four-dimensional regions the theory of relativity in the restricted sense is appropriate, if the co-ordinates are suitably chosen.

For this purpose we must choose the acceleration of the infinitely small ("local") system of co-ordinates so that no gravitational field occurs; this is possible for an infinitely small region. Let X_1, X_2, X_3, be the co-ordinates of space, and X_4 the appertaining co-ordinate of time measured in the appropriate unit.* If a rigid rod is imagined to be given as the unit measure, the co-ordinates, with a given orientation of the system of co-ordinates, have a direct physical meaning

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* The unit of time is to be chosen so that the velocity of light *in vacuo* as measured in the "local" system of co-ordinates is to be equal to unity.

in the sense of the special theory of relativity. By the special theory of relativity the expression

$$ds^2 = -dX_1^2 - dX_2^2 - dX_3^2 + dX_4^2 \quad . \quad . \quad (1)$$

then has a value which is independent of the orientation of the local system of co-ordinates, and is ascertainable by measurements of space and time. The magnitude of the linear element pertaining to points of the four-dimensional continuum in infinite proximity, we call ds. If the ds belonging to the element $dX_1 \dots dX_4$ is positive, we follow Minkowski in calling it time-like; if it is negative, we call it space-like.

To the "linear element" in question, or to the two infinitely proximate point-events, there will also correspond definite differentials $dx_1 \dots dx_4$ of the four-dimensional co-ordinates of any chosen system of reference. If this system, as well as the "local" system, is given for the region under consideration, the dX_ν will allow themselves to be represented here by definite linear homogeneous expressions of the dx_σ:—

$$dX_\nu = \sum_\sigma a_{\nu\sigma} dx_\sigma \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Inserting these expressions in (1), we obtain

$$ds^2 = \sum_{\tau\sigma} g_{\sigma\tau} dx_\sigma dx_\tau, \quad . \quad . \quad . \quad . \quad (3)$$

where the $g_{\sigma\tau}$ will be functions of the x_σ. These can no longer be dependent on the orientation and the state of motion of the "local" system of co-ordinates, for ds^2 is a quantity ascertainable by rod-clock measurement of point-events infinitely proximate in space-time, and defined independently of any particular choice of co-ordinates. The $g_{\sigma\tau}$ are to be chosen here so that $g_{\sigma\tau} = g_{\tau\sigma}$; the summation is to extend over all values of σ and τ, so that the sum consists of 4×4 terms, of which twelve are equal in pairs.

The case of the ordinary theory of relativity arises out of the case here considered, if it is possible, by reason of the particular relations of the $g_{\sigma\tau}$ in a finite region, to choose the system of reference in the finite region in such a way that the $g_{\sigma\tau}$ assume the constant values

Is GTR special in that it is the first theory in physics in which diffeo invariance is a gauge symmetry?

The Klein-Gordon equation for a scalar field Φ with mass m written in inertial coordinates (x, y, z, t) is

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} - \frac{\partial^2\Phi}{\partial t^2} - m^2\Phi = 0 \qquad (1)$$

Rewrite in generally covariant form:

$$\eta^{ab}\nabla_a\nabla_b\Phi - m^2\Phi = 0 \qquad (2)$$

Equation (2) is derivable from an action principle with

$$\mathrm{A}(\Phi, \eta) = \int \tfrac{1}{2}(\eta^{ab}\nabla_a\Phi\nabla_b\Phi + m^2\Phi^2)\sqrt{-\eta}\, d^4x \qquad (3)$$

in which Φ is varied but η_{ab} is not (it is an "absolute object"). No constraints, and therefore no non-trivial gauge.

Upshot: It seems that in special relativistic theories formal general covariance (passive diffeo invariance) obtains but active diffeo invariance as a gauge symmetry does not.

But think again! Sorkin's (2002) move. Replace the Minkowski metric η_{ab} in (2) by a general Lorentzian metric g_{ab} to get

$$g^{ab}\nabla_a\nabla_b\Phi - m^2\Phi = 0, \tag{4}$$

and add the equation

$$R_{abcd} = 0 \tag{5}$$

where R_{abcd} is the Riemann tensor computed from g_{ab}, and ∇_a is now the covariant derivative operator determined by g_{ab}. The solution sets for (2) and for (4)-(5) are the same.

To apply the constraint approach we need an action principle:

$$A(\Phi, g_{ab}, \theta^{abcd}) = \qquad (6)$$
$$\int ½(g^{ab}\nabla_a\Phi\nabla_b\Phi + m^2\Phi^2 + \theta^{abcd}R_{abcd})\,\sqrt{-g}\,d^4x$$

where the Lagrange multiplier θ^{abcd} is a tensor field with the same symmetries as the curvature tensor.

Variation with respect to θ^{abcd} gives (5).
Variation with respect to Φ gives (4).

In addition, since the metric g_{ab} is now a dynamical object, it too must be varied, with the result being an equation for θ^{abcd} that says two covariant derivatives acting on θ^{abcd} equals the stress-energy tensor for Φ.

The constrained Hamiltonian version of (4)-(6) has not been worked out, but it would be very surprising if the first class constraints did not generate phase space transformations that correspond in a natural way to the action of the spacetime diffeomorphism group.

So has active diffeo invariance as a gauge symmetry been trivialized?